Uniformly Most Powerful Test

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- Let X be a random variable with pdf $f(x, \theta)$ and we want to test the hypothesis $H_0 : \theta = \theta_0$ against $H_a : \theta \in \Omega_1$ where Ω_1 is a subset of the parameter space Ω.
- If there exists a test of H_0 which maximizes the power for any value of θ in the set of alternatives then it is said to be Uniformly Most Powerful(UMP) test for H_0 against H_a
- To obtain a UMP test for H_0 against H_a we start by testing the simple hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 \in \Omega_1$.
- Clearly the MP test exists for this test by Neyman-Pearson Lemma.
- If this test does not depend on the choice of the alternative θ_1 it is the UMP test of H_0 against all alternatives.

Example 1

Let X be normally distributed with mean μ (unknown) and variance $\sigma^2 = 1$. Test the hypothesis $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$

Solution

- The class of alternatives is $\Omega_1 = \{\mu : \mu > \mu_0\}$
- We test the hypothesis

$$H_0 : \mu = \mu_0 \ \ against \ \ \mu = \mu_1 (\mu_1 > \mu_0)$$

- The likelihood function is

$$L(X, \mu) = \prod_{i=1}^{n} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

Cont'd

- Under H_0

$$L(x, \mu_0) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_0)^2}$$

- Under H_1

$$L(x, \mu_1) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_1)^2}$$

- Then

$$\frac{L(X, \mu_1)}{L(X, \mu_0)} > k$$

is given as follows

Cont'd

$$
\begin{aligned}
\frac{L(X,\mu_1)}{L(X,\mu_0)} &= \frac{e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_1)^2}}{e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2}} > k \\
&= \frac{e^{-\frac{1}{2}\{\sum_{i=1}^{n}x_i^2-2\mu_1\sum_{i=1}^{n}x_i+n\mu_1^2\}}}{e^{-\frac{1}{2}\{\sum_{i=1}^{n}x_i^2-2\mu_1\sum_{i=1}^{n}x_i+n\mu_0^2\}}} > k \\
&= \frac{e^{\mu_1\sum_{i=1}^{n}-\frac{n}{2}\mu_1^2}}{e^{\mu_0\sum_{i=1}^{n}-\frac{n}{2}\mu_0^2}} > k \\
&= e^{\mu_1-\mu_0\sum_{i=1}^{n}x_i-\frac{n}{2}(\mu_1^2-\mu_0^2)} > k
\end{aligned}
\tag{1}
$$

Cont'd

- Taking natural logarithm we have:

$$(\mu_1 - \mu_0)\sum_{i=1}^{n} X_i - \frac{n}{2}(\mu_1^2 - \mu_0^2) > \ln k$$

$$\sum_{i=1}^{n} > \frac{\ln k + \frac{n}{2}(\mu_1^2 - \mu_0^2)}{(\mu_1 - \mu_0)}$$

$$\bar{x} > \frac{\ln k + \frac{n}{2}(\mu_1^2 - \mu_0^2)}{n(\mu_1 - \mu_0)}$$

- That is $\bar{x} > c$ So the MP test is to reject H_0 when $\bar{x} > c$ where c is such that $P(\bar{x} > c | H_0) = \alpha$

Cont'd

- But under H_0 $\bar{x} \sim N(\mu_0, \frac{1}{n})$
- Hence
$$P\left(\frac{\bar{x} - \mu_0}{\frac{1}{\sqrt{n}}} > \frac{c - \mu_0}{\frac{1}{\sqrt{n}}}\right) = \alpha$$
-
$$P(Z > \frac{c - \mu_0}{\frac{1}{\sqrt{n}}}) = \alpha \;\; where \;\; Z \sim N(0, 1)$$
- This gives
$$\frac{c - \mu_0}{\frac{1}{\sqrt{n}}} = Z_{1-\alpha}$$

Cont'd

- $$c = \frac{Z_{1-\alpha}}{\sqrt{n}} + \mu_0$$

- Therefore the critical region is

$$w = \{\bar{x} > \frac{Z_{1-\alpha}}{\sqrt{n}} + \mu_0\}$$

- We note that the MP test is independent of choice of the alternative μ_1. It is hence UMP size α test for H_0 against H_1.

Example 2

$X \sim N(0, \sigma^2)$ where σ^2 is unknown. Test the hypothesis $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma < \sigma_0$

Solution

- Modify H_1 i.e $H_1 : \sigma = \sigma_1$ and then we use Neyman-Pearson lemma.

- If $X_1, X_2, ..., X_n$ are n independent observations on X then

$$L(X,\sigma) = \left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n} X_i^2}$$

- Under H_0

$$L(X,\sigma_0) = \left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0}\sum_{i=1}^{n} X_i^2}$$

Cont'd

- Under H_1

$$L(X,\sigma_1)=\left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\sigma_1^2}\right)^{\frac{n}{2}}e^{-\frac{1}{2\sigma_1}\sum_{i=1}^{n}X_i^2}$$

- The BCR for testing H_0 against H_1 is given by:

$$\frac{L(X,\sigma_1)}{L(X,\sigma_0)}>k$$

$$\frac{L(X,\sigma_1)}{L(X,\sigma_0)}=\left(\frac{\sigma_0}{\sigma_1}\right)^{n}\frac{e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}x_i^2}}{e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}x_i^2}}>k \qquad (2)$$

Cont'd

- Taking natural logarithm we have:

$$n\ln(\frac{\sigma_0}{\sigma_1}) - \frac{1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{2\sigma_0^2}\sum_{i=1}^{n} x_i^2 > \ln k$$

- Then

$$\sum_{i=1}^{n} X_i^2\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right) > \ln k - n\ln\left(\frac{\sigma_0}{\sigma_1}\right)$$

Cont'd

- $$\sum_{i=1}^{n} x_i^2 \leq \frac{\ln k - n\ln(\frac{\sigma_0}{\sigma_1})}{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)} \;\; since \;\; \sigma_1 < \sigma_0$$

- The BCR is given as:

$$\sum_{i=1}^{n} X_i^2 \leq \frac{\ln k - n\ln(\frac{\sigma_0}{\sigma_1})}{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)} = C_\alpha$$

- $$\sum_{i=1}^{n} X_i^2 \leq C_\alpha$$

Cont'd

- Where C_α satisfies

$$
\begin{aligned}
&= P\left[\sum_{i=1}^{n} X_i^2 \leq C_\alpha | H_0\right] = \alpha \\
&= P\left[\sum_{i=1}^{n} X_i^2 \leq C_\alpha | \sigma^2 = \sigma_0^2\right] = \alpha \\
&= P\left[\frac{\sum_{i=1}^{2} X_i^2}{\sigma_0^2} \leq \frac{C_\alpha}{\sigma_0^2}\right] = \alpha
\end{aligned} \tag{3}
$$

- Where

$$\frac{C_\alpha}{\sigma_0^2} = \sigma_0^2 \chi_\alpha^2$$

Cont'd

- Then

$$\sum_{i}^{n} X_i^2 \leq \sigma_0^2 \chi_\alpha^2$$

- Since the BCR does not depend on the particular value of σ_1 it follows that the UMP size $\alpha-$ test reject the H_0 whenever

$$\sum_{i}^{n} X_i^2 \leq \sigma_0^2 \chi_\alpha^2$$

Thank You!